University of Toronto Math Academy
Qualifying Quiz

Instructions

We call it a quiz, but it's really a challenge: a chance for you to show us how you approach new problems and new concepts in mathematics. We care about more than just your final answers—your reasoning, your explanations, and your presentation are just as important.

Some of the problems are intended to be hard. We do not expect you to solve every problem completely. Send us your work, even if it is only partial solutions or conjectures. There is no time limit on this quiz.

If you need clarification on a problem, please email `outreach@math.toronto.edu`. You may not consult or get help from anyone else.

We also think that these problems are fun. So, enjoy, and good luck!

The problems

1. At the University of Toronto Math Academy, each student always tells the truth or always lies. One day there are 21 students in class and every student says "Everybody else is a liar". What is the largest possible number of truthtellers? What is the largest possible number of liars?

2. Which integer numbers can be written as the difference of two perfect squares?

   Your answer should be a list of *all* the integer numbers that can be written as the difference of two perfect squares. Once you have that, you still have to do two things. First, prove that all the numbers in your list are, in fact, the difference of two squares. Second, prove that no other integer numbers are the difference of two perfect squares.

3. I have a collection of chameleons in my garden. Each chameleon is either red, green, or black. The chameleons sometimes change colours: if two chameleons of different colours run into each other, they both change to the third colour. This is the only type of colour change that may happen. Yesterday I noticed that I had 4 green chameleons, 5 red chameleons, and 6 black chameleons. Prove that my collection of chameleons will never be monochromatic (that is, all the same colour).

4. Rey and Kylo Ren are playing a game. They have a big bag of M&Ms of various colours and they take turns eating them. In their turn, they can choose to eat one or two M&Ms, but they can only eat two if they are of different colours. Rey will

go first. The person who eats the last M&M wins the game. Of course, the outcome will depend on how many M&Ms of each colour there were at the beginning.

(a) Assume they start with 4 red M&Ms and 3 green M&Ms. If both players play perfectly, who will win?

(b) If we start with any number of red M&Ms and green M&Ms, what is the winning strategy?

(c) What if they have M&Ms of three different colours instead?

(d) Bonus question: Can you find a winning strategy for four different colours? For five?
Note: We have been able to completely solve this game for up to five colours, but we do not know what the winning strategy with six colours is. If you figure it out, we would love to hear it!

5. I have a tetrahedron. (If you do not know what a tetrahedron is, Google it.) I inscribe a blue sphere inside the tetrahedron (this means that the sphere is inside the tetrahedron and the sphere touches all four faces of the tetrahedron). I also inscribe the tetrahedron inside a yellow sphere (this means that the tetrahedron is inside the sphere and the sphere touches all four vertices of the tetrahedron). Calculate the ratio of the radius of the blue sphere to the radius of the yellow sphere.